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Subdifferentials of perturbed distance functions in Banach spaces

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Abstract In general Banach space setting, we study the perturbed distance function $d_S^J(\cdot)$ determined by a closed subset S and a lower semicontinuous function $J(\cdot)$. In particular, we show that the Fréchet subdifferential and the proximal subdifferential of a perturbed distance function are representable by virtue of corresponding normal cones of S and subdifferentials of $J(\cdot)$.

Keywords Subdifferential · Fréchet subdifferential · Proximal subdifferential · Perturbed optimization problem · Well-posedness

Mathematics Subject Classification (2000) 49J52 · 46N10 · 49K27

1 Introduction

Let X be a real Banach space endowed with norm $\|\cdot\|$ and let S be a nonempty closed subset of X. Let $J : S \to \mathbb{R}$ be a lower semicontinuous function. We define the perturbed distance

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function $d_S^J : X \to \mathbb{R}$ by

$$d_S^J(x) = \inf_{s \in S}\{\|x - s\| + J(s)\} \quad \text{for each } x \in X. \tag{1.1}$$

For $x \in X$, the perturbed optimization problem is to find an element $z_0 \in S$ such that

$$\|x - z_0\| + J(z_0) = d_S^J(x) \tag{1.2}$$

which is denoted by $\min_J(x, S)$. Any point z_0 satisfying (1.2) (if it exists) is called a solution of the problem $\min_J(x, S)$. In particular, if $J \equiv 0$, then the perturbed distance function d_S^J and the perturbed optimization problem $\min_J(x, S)$ reduce to distance function d_S and the well-known best approximation problem, respectively.

The perturbed optimization problem $\min_J(x, S)$ was presented and investigated by Baranger in [1] and Bidaut in [5]. The existence results have been applied to optimal control problems governed by partial differential equations, see for example, [1–5,14,18,27]. Under the assumption that J is bounded from below, Baranger proved in [1] that if X is a uniformly convex Banach space then the set of all $x \in X$ for which the problem $\min_J(x, S)$ has a solution is a dense G_δ-subset of X, which extends Stechkin's results in [30] on the best approximation problem. Since then, this problem has been extensively studied, see for example [5,14,22,23,29]. In particular, Cobzas [15] extended Baranger's result to the setting of reflexive Kadec Banach space; while Ni [28] relaxed the reflexivity assumption made in Cobzas' result.

Distance functions play an important role in optimization and variational analysis (see [7,8,10,24,25]). For example, distance functions are fundamental to multiplier existence theorems in constrained optimization [8], and algorithms for solving nonlinear systems of equations and nonlinear programs (see [7,10]). In general, distance functions of nonempty closed subsets in Banach spaces are nonconvex and so the study of various subdifferentials of distance functions have received a lot of attention (see [6,9,12,13,16,17,31]). In particular, Burke et al [9] developed the Clarke subdifferentials of distance functions in terms of corresponding normal cones of associated subsets and the similar result for the Fréchet subdifferentials is due to Kruger [21] and Ioffe [20] (see also [6]). The proximal subdifferentials of distance functions are presented in [6] in terms of corresponding normal cones of the associated subsets. Extensions of these results to the setting of a minimal time function determined by a closed convex set and a closed set have been done recently, see for example [19,26] and references therein.

The purpose of this paper is to explore both the Fréchet subdifferentials and the proximal subdifferentials of perturbed distance functions $d_S^J(\cdot)$. Our main results extend the corresponding ones in [6,9,24,25] from distance functions to general perturbed distance functions.

2 Preliminaries

Let X be a normed vector space with norm denoted by $\|\cdot\|$. Let X^* denote the topological dual of X. We use $\boldsymbol{B}(x, r)$ to denote the open ball centered at x with radius $r > 0$, and $\langle \cdot, \cdot \rangle$ to denote the pairing between X^* and X. Let $\mathbb{B}$ (resp. $\mathbb{B}^*$) denote the closed unit ball of

X (resp. X^*) centered at the origin. Let S be a nonempty closed subset of X. We use δ_S to denote the indicator function of S, i.e.,

$$\delta_S(x) = \begin{cases} 0 & x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

Write $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and let $f : X \to \mathbb{R}$ be a proper *lower semicontinuous* (l.s.c.) function. The effective domain of f is denoted by

$$\mathrm{D}(f) = \{x \in X \mid f(x) < +\infty\}.$$

The notions in the following definition are well-known (see for example [6,20,21]).

Definition 2.1 Let $x \in \mathrm{D}(f)$.

(1) The Fréchet subdifferential $\partial^F f(x)$ of f at x is defined by

$$\partial^F f(x) = \left\{ x^* \in X^* \,\middle|\, \liminf_{y \to x} \frac{f(y) - f(x) - \langle x^*, y - x\rangle}{\|y - x\|} \geq 0 \right\}.$$

(2) The proximal subdifferential $\partial^P f(x)$ of f at x is defined by

$$\partial^P f(x) = \left\{ x^* \in X^* \,\middle|\, \liminf_{y \to x} \frac{f(y) - f(x) - \langle x^*, y - x\rangle}{\|y - x\|^2} > -\infty \right\}.$$

Let $x \in D(f)$ Clearly, an element $x^* \in \partial^F f(x)$ if and only if, for any $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \varepsilon\|y - x\| \quad \text{for all } y \in \boldsymbol{B}(x, \rho). \tag{2.1}$$

It is also clear that $x^* \in \partial^P f(x)$ if and only if, there exist $\rho, \sigma > 0$ such that

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + \sigma\|y - x\|^2 \quad \text{for all } y \in \boldsymbol{B}(x, \rho). \tag{2.2}$$

Hence we have that $\partial^P f(x) \subset \partial^F f(x)$. Furthermore, in the case when f is convex, we have that

$$\partial^P f(x) = \partial^F f(x) = \partial f(x) \quad \text{for each } x \in \mathrm{D}(f),$$

where $\partial f(x)$ is the subdifferential of f at x (in the sense of convex analysis) defined by

$$\partial f(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x), \ \forall y \in X\}.$$

In particular, for $x \in S$, the Fréchet subdifferential and the proximal subdifferential of the indicator function of S at x are called the *Fréchet normal cone* and respectively the *proximal normal cone* of S at x, i.e.,

$$N_S^F(x) = \partial^F \delta_S(x) \quad \text{and} \quad N_S^P(x) = \partial^P \delta_S(x).$$

In the case when S is convex, the Fréchet normal cone and the proximal normal cone of S at x coincide with the normal cone $N_S(x)$ of S at x, which is defined by

$$N_S(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq 0, \ \forall y \in S\}.$$

Moreover, by definitions, the assertion in the following remark is direct.

Remark 2.1 Let $f : X \to \mathbb{R}$ be a Lipschitz continuous function with modulus $L > 0$. Then we have

$$\partial^P f(x) \subseteq L\mathbb{B}^* \quad \text{and} \quad \partial^F f(x) \subseteq L\mathbb{B}^* \quad \text{for each } x \in X. \tag{2.3}$$

3 Subdifferentials of perturbed distance functions

Note that J is only defined on S. For the whole section, we make the following definition:

$$(J + \delta_S)(x) = \begin{cases} J(x) & x \in S \\ +\infty & \text{otherwise.} \end{cases}$$

Recall that the perturbed distance function is defined by

$$d_S^J(x) = \inf_{s \in S}\{\|x - s\| + J(s)\} \quad \text{for each } x \in X.$$

By [29] we have that

$$|d_S^J(y) - d_S^J(x)| \leq \|y - x\| \quad \text{for any} x, y \in X. \tag{3.1}$$

This and (2.3) imply that

$$\partial^P d_S^J(x) \subseteq \mathbb{B}^* \quad \text{and} \quad \partial^F d_S^J(x) \subseteq \mathbb{B}^* \quad \text{for each } x \in X. \tag{3.2}$$

In particular, if S is convex, then we have

$$\partial d_S^J(x) \subseteq \mathbb{B}^*. \tag{3.3}$$

Following [23,29], let S_0 denote the set of all points $x \in S$ such that x is a solution of the problem $\min_J(x, S)$, i.e.,

$$S_0 = \{x \in S |\, d_S^J(x) = J(x)\}. \tag{3.4}$$

For the remainder of the present paper, we assume that S_0 is non-empty.

3.1 The convex case

In this subsection we always assume that S is a closed convex subset of X and $J : S \to \mathbb{R}$ is convex. Then it is easy to verify that $d_S^J(\cdot)$ is convex on X. The main theorem of this subsection is as follows.

Theorem 3.1 *Suppose that $S \subset X$ is closed convex and $J : S \to \mathbb{R}$ is convex. The following assertions hold:*

(1) *If $x \in S$, then we have*

$$\partial d_S^J(x) \supseteq \partial(J + \delta_S)(x) \cap \mathbb{B}^*. \tag{3.5}$$

(2) *If $x \in S_0$, then we have*

$$\partial d_S^J(x) = \partial(J + \delta_S)(x) \cap \mathbb{B}^*. \tag{3.6}$$

Proof

(1) Let x^* belong to the set on the right-hand side of (3.5). Then $\|x^*\| \le 1$ and

$$\langle x^*, s - x\rangle \le J(s) - J(x) \quad \text{for each } s \in S. \tag{3.7}$$

It follows that

$$\|y - s\| \ge \langle x^*, y - s\rangle = \langle x^*, y - x\rangle - \langle x^*, s - x\rangle \tag{3.8}$$

holds for each $y \in X$ and $s \in S$. Hence, we have from (3.7) and (3.8) that

$$\|y - s\| + J(s) \ge \langle x^*, y - x\rangle + J(x) \quad \text{for each } y \in X \text{ and } s \in S. \tag{3.9}$$

Since $d_S^J(x) \le J(x)$ (as $x \in S$), it follows that

$$d_S^J(y) - d_S^J(x) \ge \inf_{s\in S}\{\|y - s\| + J(s)\} - J(x) \quad \text{for each } y \in X. \tag{3.10}$$

This together with (3.9) implies that

$$\begin{aligned} d_S^J(y) - d_S^J(x) &\ge \inf_{s\in S}\{\|y - s\| + J(s)\} - J(x) \\ &\ge \inf_{s\in S}\{\langle x^*, y - x\rangle + J(x)\} - J(x) \\ &= \langle x^*, y - x\rangle. \end{aligned}$$

Hence, $x^* \in \partial d_S^J(x)$ and (3.5) is proved.

(2) Let $x \in S_0$. By (3.3), one sees that $\partial d_S^J(x) \subset \mathbb{B}^*$. Hence we only need to prove that

$$\partial d_S^J(x) \subseteq \partial(J + \delta_S)(x). \tag{3.11}$$

To do this, let $x^* \in \partial d_S^J(x)$. Then, for each $y \in X$ we have

$$\langle x^*, y - x\rangle \le d_S^J(y) - d_S^J(x) = \inf_{s\in S}\{\|y - s\| + J(s)\} - J(x), \tag{3.12}$$

where the equality holds because of (3.4). Thus, for each $y \in S$ we have that

$$\langle x^*, y - x\rangle \le J(y) - J(x) = J(y) + \delta_S(y) - (J(x) + \delta_S(x)). \tag{3.13}$$

Note that (3.13) is trivial for any $y \notin S$. Hence $x^* \in \partial(J + \delta_S)(x)$ and the proof of (3.11) is complete. □

In particular, if $J \equiv 0$, then $S = S_0$. Thus, from Theorem 3.1, we get the following corollary, which was shown in [9,11,24,25].

Corollary 3.1 *Let* $J \equiv 0$. *There holds*

$$\partial d_S(x) = N_S(x) \cap \mathbb{B}^* \quad \textit{for each } x \in S.$$

3.2 The nonconvex case

This subsection is devoted to the study of the Fréchet subdifferential and the proximal subdifferential for the case when S is nonconvex. To this end we first introduce the notion of the well-posedness (cf. [23]). Recall that a sequence $\{z_n\} \subseteq S$ is a *minimizing sequence* of the problem $\min_J(x, S)$ if

$$\lim_{n\to+\infty}(\|x - z_n\| + J(z_n)) = \inf_{z\in S}(\|x - z\| + J(z)).$$

Recall also that the problem $\min_J(x, S)$ is *well-posed* if $\min_J(x, S)$ has a unique solution and every minimizing sequence of the problem $\min_J(x, S)$ converges to this solution. Before starting our main theorems, we need some lemmas.

Lemma 3.1 *Let* $\varepsilon > 0$, $\rho > 0$, *and let* $x \in S_0$. *Suppose that* $\min_J(x, S)$ *is well-posed. Then there exists* $r \in (0, 1)$ *such that for any* $z \in \boldsymbol{B}(x, r)$ *and* $y \in S$ *if*

$$\|z - y\| + J(y) \le d_S^J(z) + \varepsilon\|z - x\| \tag{3.14}$$

holds, then we have

$$\|y - x\| < \rho. \tag{3.15}$$

Proof We proceed by contradiction. Suppose on the contrary that, for each $n = 1, 2, \cdots$, there exist $z_n \in \boldsymbol{B}(x, \frac{1}{n})$ and $y_n \in S$ such that

$$\|z_n - y_n\| + J(y_n) \le d_S^J(z_n) + \varepsilon\|z_n - x\| \tag{3.16}$$

and

$$\|y_n - x\| \ge \rho. \tag{3.17}$$

Thus, we have $\lim_{n\to+\infty}\|z_n - x\| = 0$. Note that

$$d_S^J(z_n) \le \|x - y_n\| + J(y_n) \le \|x - z_n\| + \|z_n - y_n\| + J(y_n).$$

This together with (3.16) implies that

$$\begin{aligned} d_S^J(z_n) &\le \|x - y_n\| + J(y_n) \\ &\le \|x - z_n\| + d_S^J(z_n) + \varepsilon\|z_n - x\| \\ &= d_S^J(z_n) + (1 + \varepsilon)\|z_n - x\|. \end{aligned} \tag{3.18}$$

Since $d_S^J(\cdot)$ is continuous, it follows that

$$d_S^J(x) \le \lim_{n\to+\infty}(\|x - y_n\| + J(y_n)) \le d_S^J(x), \tag{3.19}$$

i.e., $\{y_n\}$ is a *minimizing sequence* of the problem $\min_J(x, S)$. Noting that $x \in S_0$ and $\min_J(x, S)$ is well-posed, we obtain

$$\lim_{n\to+\infty} y_n = x,$$

which contradicts (3.17). This completes the proof of the lemma. □

Lemma 3.2 *Let* $\varepsilon > 0$. *Let* $x \in S_0$, *and let* $y \in X$ *be such that*

$$\langle x^*, y - x\rangle \le d_S^J(y) - d_S^J(x) + \varepsilon\|y - x\|. \tag{3.20}$$

Then we have

$$\langle x^*, y - x\rangle \le J(y) + \delta_S(y) - (J(x) + \delta_S(x)) + \varepsilon\|y - x\|, \tag{3.21}$$

Proof Let $y \in X$. Since (3.21) is trivial if $y \notin S$, we may assume that $y \in S$. As $x \in S_0$, we have from (3.4) that

$$d_S^J(y) - d_S^J(x) + \varepsilon\|y - x\| = \inf_{s\in S}\{\|y - s\| + J(s)\} - J(x) + \varepsilon\|y - x\|.$$

Combining this with (3.20) gives that

$$\begin{aligned}\langle x^*, y - x\rangle &\leq \inf_{s\in S}\{\|y - s\| + J(s)\} - J(x) + \varepsilon\|y - x\| \\ &\leq J(y) - J(x) + \varepsilon\|y - x\| \\ &= J(y) + \delta_S(y) - (J(x) + \delta_S(x)) + \varepsilon\|y - x\|,\end{aligned}$$

which shows (3.21) and completes the proof. □

Lemma 3.3 *Let $L, \varepsilon_1, \varepsilon_2 \in (0, 1)$. Let $x \in S_0,\ z,\ y \in X$, and let $x^* \in \mathbb{B}^*$ be such that*

$$\langle x^*, y - x\rangle \leq J(y) - J(x) + \varepsilon_1\|y - x\|, \tag{3.22}$$

$$\|z - y\| + J(y) \leq d_S^J(z) + \varepsilon_2\|z - x\| \tag{3.23}$$

and

$$|J(y) - J(x)| \leq L\|y - x\|. \tag{3.24}$$

Then we have

$$\|y - x\| < \frac{3}{1 - L}\|z - x\| \tag{3.25}$$

and

$$\langle x^*, z - x\rangle \leq d_S^J(z) - d_S^J(x) + \left(\varepsilon_2 + \frac{3\varepsilon_1}{1 - L}\right)\|z - x\|. \tag{3.26}$$

Proof By (3.23) and the definition of $d_S^J(z)$ we have

$$\begin{aligned}\|y - x\| &\leq \|y - z\| + \|z - x\| \\ &\leq d_S^J(z) - J(y) + \varepsilon_2\|z - x\| + \|z - x\| \\ &\leq J(x) - J(y) + (2 + \varepsilon_2)\|z - x\|.\end{aligned}$$

Using (3.24), we get that

$$\|y - x\| \leq L\|y - x\| + (2 + \varepsilon_2)\|z - x\|.$$

Hence,

$$\|y - x\| \leq \frac{\varepsilon_2 + 2}{1 - L}\|z - x\| < \frac{3}{1 - L}\|z - x\|, \tag{3.27}$$

which implies that (3.25) holds. To show (3.26), we use (3.22) and (3.27) to conclude that

$$\begin{aligned}\langle x^*, y - x\rangle &\leq J(y) - J(x) + \varepsilon_1\|y - x\| \\ &\leq J(y) - J(x) + \tfrac{3\varepsilon_1}{1 - L}\|z - x\|.\end{aligned} \tag{3.28}$$

Since $\|x^*\| \leq 1$, it follows from (3.23) that

$$\langle x^*, z - y\rangle \leq \|z - y\| \leq d_S^J(z) - J(y) + \varepsilon_2\|z - x\|. \tag{3.29}$$

This together with (3.28) implies that

$$\begin{aligned}\langle x^*, z - x\rangle &= \langle x^*, z - y\rangle + \langle x^*, y - x\rangle \\ &\le d_S^J(z) - J(y) + \varepsilon_2\|z - x\| + J(y) - J(x) + \tfrac{3\varepsilon_1}{1-L}\|\|z - x\| \\ &= d_S^J(z) - d_S^J(x) + \left(\varepsilon_2 + \tfrac{3\varepsilon_1}{1-L}\right)\|z - x\|,\end{aligned} \tag{3.30}$$

where the last equality holds because of (3.4). Thus, inequality (3.26) holds, which completes the proof of the lemma. □

Recall that J satisfies *the center locally Lipschitz condition at x* if there exist $\rho > 0$ and $L > 0$ such that

$$|J(y) - J(x)| \le L\|y - x\| \quad \text{for each } y \in \boldsymbol{B}(x, \rho).$$

We define *center locally Lipschitz constant $L_x \in [0, +\infty]$ at x* by

$$L_x = \inf_{\rho>0} \sup_{y\in \boldsymbol{B}(x,\rho)} \frac{|J(y) - J(x)|}{\|y - x\|}.$$

Then, it is clear that J satisfies center locally Lipschitz condition at x if and only if $L_x < +\infty$.

Theorem 3.2 *Let $x \in S_0$. The following assertions hold.*

(1) *We have*

$$\partial^F d_S^J(x) \subset \partial^F(J + \delta_S)(x) \cap \mathbb{B}^*. \tag{3.31}$$

(2) *If* $\min_J(x, S)$ *is well-posed and* $L_x < 1$, *then we have*

$$\partial^F d_S^J(x) = \partial^F(J + \delta_S)(x) \cap \mathbb{B}^*. \tag{3.32}$$

Proof

(1) Let $x^* \in \partial^F d_S^J(x)$. Then $x^* \in \mathbb{B}^*$ by (3.2). Thus, to prove (3.31), it is sufficient to prove that $x^* \in \partial^F(J + \delta_S)(x)$. To proceed, let $\varepsilon > 0$. Then there exists $\rho > 0$ such that

$$\langle x^*, y - x\rangle \le d_S^J(y) - d_S^J(x) + \varepsilon\|y - x\| \quad \text{for each } y \in \boldsymbol{B}(x, \rho). \tag{3.33}$$

Thus, by Lemma 3.2, it follows that

$$\langle x^*, y - x\rangle \le J(y) + \delta_S(y) - (J(x) + \delta_S(x)) + \varepsilon\|y - x\| \quad \text{for each} y \in \boldsymbol{B}(x, \rho),$$

which implies that $x^* \in \partial^F(J + \delta_S)(x)$. Hence, condition (3.31) holds.

(2) Assume that $\min_J(x, S)$ is well-posed and $L_x < 1$. To show (3.32), it is sufficient to prove that

$$\partial^F(J + \delta_S)(x) \cap \mathbb{B}^* \subset \partial^F d_S^J(x). \tag{3.34}$$

To proceed, take $x^* \in \partial^F(J + \delta_S)(x)$ be such that $\|x^*\| \le 1$. Let $\varepsilon > 0$, and let $L_x < L < 1$. Set

$$\varepsilon_0 = \min\left\{1, \frac{(1 - L)\varepsilon}{4 - L}\right\}.$$

Then, $\varepsilon_0 > 0$ because $L < 1$. Moreover, there exists $\rho > 0$ satisfying

$$\langle x^*, y - x\rangle \le J(y) - J(x) + \varepsilon_0\|y - x\| \quad \text{for each } y \in S \cap \boldsymbol{B}(x, \rho) \tag{3.35}$$

and

$$|J(y) - J(x)| \le L\|y - x\| \quad \text{for each } y \in \boldsymbol{B}(x, \rho). \tag{3.36}$$

By Lemma 3.1 (applied to ε_0 in place of ε), there exists $r \in (0, 1)$ such that for each $z \in B(x, r)$ and $y \in S$, one has hat

$$\|z - y\| + J(y) \le d_S^J(z) + \varepsilon_0\|z - x\| \Longrightarrow \|y - x\| < \rho. \tag{3.37}$$

Take $z \in \boldsymbol{B}(x, r) \setminus \{x\}$. By the definition of $d_S^J(z)$, we can take $y_z \in S$ such that

$$\|z - y_z\| + J(y_z) \le d_S^J(z) + \varepsilon_0\|z - x\|. \tag{3.38}$$

By (3.37), we have

$$\|y_z - x\| < \rho. \tag{3.39}$$

Combining this with (3.35) and (3.36) yields that

$$\langle x^*, y_z - x\rangle \le J(y_z) - J(x) + \varepsilon_0\|y_z - x\| \tag{3.40}$$

and

$$|J(y_z) - J(x)| \le L\|y_z - x\|. \tag{3.41}$$

Then assumptions in Lemma 3.3 are satisfied with $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$ and $y = y_z$, and Lemma 3.3 allows us to conclude that

$$\begin{aligned}\langle x^*, z - x\rangle &\le d_S^J(z) - d_S^J(x) + \left(\varepsilon_0 + \tfrac{3\varepsilon_0}{1-L}\right)\|z - x\| \\ &\le d_S^J(z) - d_S^J(x) + \varepsilon\|z - x\|,\end{aligned} \tag{3.42}$$

which shows that $x^* \in \partial^F d_S^J(x)$ as $\epsilon > 0$ is arbitrary. Thus, (3.34) is proved and the proof of the theorem is complete. □

Theorem 3.3 *Let $x \in S_0$. The following assertions hold.*

(1) *We have*

$$\partial^P d_S^J(x) \subset \partial^P (J + \delta_S)(x) \cap \mathbb{B}^*. \tag{3.43}$$

(2) *If* $\min_J(x, S)$ *is well-posed and* $L_x < 1$, *then we have*

$$\partial^P d_S^J(x) = \partial^P (J + \delta_S)(x) \cap \mathbb{B}^*. \tag{3.44}$$

Proof

(1) Let $x^* \in \partial^P d_S^J(x)$. Then, there exist two positive numbers $\sigma, \rho > 0$ such that

$$\langle x^*, y - x\rangle \le d_S^J(y) - d_S^J(x) + \sigma\|y - x\|^2 \quad \text{for each} y \in \boldsymbol{B}(x, \rho). \tag{3.45}$$

Let $y \in \boldsymbol{B}(x, \rho)$ and set $\varepsilon = \sigma\|y - x\|$. Thus, (3.45) implies that (3.20) in Lemma 3.2 holds. Hence, we can use Lemma 3.2 to conclude that

$$\begin{aligned}\langle x^*, y - x\rangle &\le J(y) + \delta_S(y) - (J(x) + \delta_S(x)) + \varepsilon\|y - x\| \\ &= J(y) + \delta_S(y) - (J(x) + \delta_S(x)) + \sigma\|y - x\|^2,\end{aligned}$$

which implies that $x^* \in \partial^P(J + \delta_S)(x)$. Note that $x^* \in \mathbb{B}^*$ follows directly from (3.2). Hence, $x^* \in \partial^P(J + \delta_S)(x) \cap \mathbb{B}^*$ and we arrive at (3.43).

(2) Assume that $\min_J(x, S)$ is well-posed and $L_x < 1$. Let $x^* \in \partial^P(J + \delta_S)(x) \cap \mathbb{B}^*$, and let $L_x < L < 1$. Then $\|x^*\| \le 1$ and there exist two postive numbers $\rho, \sigma > 0$ such that

$$\langle x^*, y - x\rangle \le J(y) - J(x) + \sigma\|y - x\|^2 \quad \text{for each } y \in S \cap \boldsymbol{B}(x, \rho) \tag{3.46}$$

and

$$|J(y) - J(x)| \le L\|y - x\| \quad \text{for each } y \in \boldsymbol{B}(x, \rho). \tag{3.47}$$

By Lemma 3.1 (with $\varepsilon = 1$), there exists $0 < r < 1$ such that for each $z \in B(x, r)$ and $y \in S$

$$\|z - y\| + J(y) \le d_S^J(z) + \|z - x\| \Longrightarrow \|y - x\| < \rho. \tag{3.48}$$

Let $z \in \boldsymbol{B}(x, r) \setminus \{x\}$. In view of the definition of $d_S^J(z)$, one can take $y_z \in S$ such that

$$\|z - y_z\| + J(y_z) \le d_S^J(z) + \|z - x\|^2. \tag{3.49}$$

Hence

$$\|z - y_z\| + J(y_z) \le d_S^J(z) + \|z - x\|. \tag{3.50}$$

By (3.48), we have

$$\|y_z - x\| < \rho. \tag{3.51}$$

This, together with (3.46) and (3.47), implies that

$$\langle x^*, y_z - x\rangle \le J(y_z) - J(x) + \sigma\|y_z - x\|^2 \tag{3.52}$$

and

$$|J(y_z) - J(x)| \le L\|y_z - x\|. \tag{3.53}$$

Write

$$\varepsilon_1 = \sigma\|y_z - x\| \quad \text{and} \quad \varepsilon_2 = \|z - x\|. \tag{3.54}$$

Thus, (3.49), (3.52) and (3.53) imply that assumptions in Lemma 3.3 are satisfied with $\varepsilon_1, \varepsilon_2$ given by (3.54) and $y = y_z$. Hence, Lemma 3.3 allows us to conclude that

$$\|y_z - x\| < \frac{3}{1 - L}\|z - x\| \tag{3.55}$$

and

$$\langle x^*, z - x\rangle \le d_S^J(z) - d_S^J(x) + \left(\varepsilon_2 + \frac{3\varepsilon_1}{1 - L}\right)\|z - x\|, \tag{3.56}$$

which together with (3.54) yields that

$$\langle x^*, z - x\rangle \le d_S^J(z) - d_S^J(x) + \left(1 + \frac{3\sigma}{(1 - L)^2}\right)\|z - x\|^2.$$

Hence, $x^* \in \partial^P d_S^J(x)$ and so we arrive at $\partial^P(J + \delta_S)(x) \cap \mathbb{B}^* \subset \partial^P d_S^J(x)$. This completes the proof of the theorem. □

Lemma 3.4 *Let $x \in S_0$. Assume that $J(\cdot)$ satisfies the center Lipschitz condition on S at x with Lipschitz constant $0 \le L < 1$, i.e.,*

$$\|J(y) - J(x)\| \le L\|y - x\| \quad \text{for each } y \in S. \tag{3.57}$$

Then $\min_J(x, S)$ *is well-posed.*

Proof Since $x \in S_0$, x is a solution of $\min_J(x, S)$. Below, we show that every minimizing sequence of $\min_J(x, S)$ converges to x. Granting this, x is the unique solution of $\min_J(x, S)$; hence $\min_J(x, S)$ is well-posed. To proceed, let $\{z_n\} \subset S$ be a minimizing sequence of $\min_J(x, S)$, i.e.,

$$\lim_{n\to+\infty}(\|x - z_n\| + J(z_n)) = \inf_{s\in S}\{\|x - s\| + J(s)\} = J(x).$$

Thus for each $\varepsilon > 0$ there exists a positive integer N such that if $n \ge N$, then

$$\|x - z_n\| + J(z_n) \le J(x) + \varepsilon.$$

This, together with (3.57), gives that

$$\|x - z_n\| \le J(x) - J(z_n) + \varepsilon \le L\|x - z_n\| + \varepsilon,$$

which implies that

$$\|x - z_n\| \le \frac{\varepsilon}{1 - L}.$$

Consequently we arrive at

$$\lim_{n\to+\infty}\|x - z_n\| = 0.$$

This completes the proof of the lemma. □

By Lemma 3.4, the next corollary follows directly from Theorems 3.2 and 3.3.

Corollary 3.2 *Let $x \in S_0$. Then we have*

$$\partial^F d_S^J(x) \subset \partial^F(J + \delta_S)(x) \cap \mathbb{B}^* \quad \text{and} \quad \partial^P d_S^J(x) \subset \partial^P(J + \delta_S)(x) \cap \mathbb{B}^*.$$

Furthermore, if $J(\cdot)$ satisfies the center Lipschitz constant at x with Lipschitz constant $0 \le L < 1$, then we have

$$\partial^F d_S^J(x) = \partial^F(J + \delta_S)(x) \cap \mathbb{B}^* \quad \text{and} \quad \partial^P d_S^J(x) = \partial^P(J + \delta_S)(x) \cap \mathbb{B}^*.$$

In particular, letting $J \equiv 0$, we get the following corollary, which was proved in [6].

Corollary 3.3 *Let $x \in S$. Then we have*

$$\partial^F d_S(x) = N_S^F(x) \cap \mathbb{B}^* \quad \text{and} \quad \partial^P d_S(x) = N_S^P(x) \cap \mathbb{B}^*.$$

We end this paper with a remark about Lipschitz conditions.

Remark 3.1 Recall that a function $J : X \to \mathbb{R}$ satisfies *the locally Lipschitz condition at x* if there exist $L > 0$ and $\rho > 0$ such that

$$|J(y) - J(z)| \leq L\|y - z\| \quad \text{for each } y,\ z \in \boldsymbol{B}(x,\ \rho).$$

Obviously, the locally Lipschitz condition implies the center locally Lipschitz condition. However, the converse is not true, in general, as shown in the following example.

Example 3.1 Let $X = \mathbb{R}$, and let $J : X \to \mathbb{R}$ be defined by

$$J(x) = \begin{cases} \frac{1}{2}x \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Let $x = 0$. Then $J(\cdot)$ satisfies the center Lipschitz condition on X at $x = 0$ with Lipschitz constant $L = \frac{1}{2}$ but it does not satisfy the locally Lipschitz condition at $x = 0$.

In fact, take $y_k = \frac{1}{k}$ and $x_k = \frac{2}{2k+1}$ for each $k = 1, 2, \cdots$. Then

$$|J(y_k) - J(x_k)| = \frac{1}{2}\left|y_k \sin\frac{\pi}{y_k} - x_k \sin\frac{\pi}{x_k}\right| = \frac{1}{2k + 1}.$$

Thus, we get

$$\frac{|J(y_k) - J(x_k)|}{|y_k - x_k|} = k,$$

which implies that $J(\cdot)$ dose not satisfy the local Lipschitz condition at $x = 0$. Note that for each $y \in X$ and $y \neq 0$ we have

$$|J(y) - J(0)| = \frac{1}{2}\left|y \sin\frac{\pi}{y}\right| \leq \frac{1}{2}\|y - 0\|,$$

which shows that $J(\cdot)$ satisfies the center Lipschitz condition on X at $x = 0$ with Lipschitz constant $L = \frac{1}{2}$.

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